THE BLOKKOLOGY PROBLEM: Proof in words

(*The previous proof contained a small gap, in failing to address collections of 5 tiles such that any square in a row can be reached using 2 of those tiles. Our conclusion remains the same – there is no collection of 5 tiles that can span the entire Blokkology board from an arbitrary starting square – but the math ends up being far more complicated than we realized at first. Our claims in the second half of this proof were arrived at by rather laborious pencil-and-paper lists. If someone out there can form a more elegant theory out of all this, we would be interested and grateful.)*

Moving from a starting square to an arbitrary ending square is equivalent to moving some distance horizontally, then vertically (or vice versa).

So, being able to move to *any* other square on the board is equivalent to being able to both:

- make a horizontal move to *any* square in your current row, AND
- make a vertical move to *any* square in your current column.

For example, using familiar algebraic notation as in chess, moving from b3 to f1 is equivalent to moving 4 squares right and 2 squares down, in either order.

So we first break the problem down into just the horizontal component: how many tiles are required to be able to move to *any* of the 10 squares in your current row? (Here we count your current square as one of those 10, even though obviously it doesn't take any tiles to stay where you are.)

Well, for each of the tiles you have, you either:

- use it to move right;
- use it to move left; or
- don't use it at all.

That means each tile has 3 possible uses. Since the order of the tiles doesn't matter in this problem, we can easily figure out how many horizontal moves can be made using some or all of our tiles. The total number of combinations is:

(3 possible uses for the first tile) x (3 uses for the second tile) x … x (3 uses for the last tile), or $\mathbf{3^n}$, where *n* is the number of tiles you have.

(Note: not all of these moves may be distinct from each other – e.g., with tiles 1, 3, and 5, two of those combinations [1 left, 5 right, AND 1 right, 3 right] produce the same move, namely 4 right. Thus 3^n is really the *maximum* number – an "upper bound" – for *distinct* horizontal moves.)

Let's say we have two tiles ($n = 2$). Then we get 3^2 or 9 possible horizontal moves (at *most*, based on our note above). One of those 9, though, is the combination where we don't use either tile at all, so really there are at most **8** moves that actually take us to a *different* square in our row. But since a Blokkology board is 10-by-10, there are **9** other squares in our row! Therefore, it's clear that a 2-tile set cannot take us to *any* square in our row – there must be at least one square that is unreachable. (In fact, many more than one—see Appendix.)

The same argument holds for the vertical component of our move: 3 tiles are also required to be able to move to *any* square in a given *column*, or equivalently, given any 2 tiles, we can come up with a number of squares that we can*not* move up/down.

At this point, it's useful to know exactly *which* 3-tile collections allow us to move to any square in a given row or column. As it turns out – based off about an hour's worth of double-checking – there are only 6 of these triplets that work: (1, 2, 6), (1, 2, 7), (1, 3, 5), (1, 3, 6), (1, 3, 7), and (3, 4, 5).

NB: In Blokkology, it is *not* sufficient that the tiles are able to add/subtract to any number between 0-9; we also have to make sure that tiles never carry us off the board! For example, the false triplet (1, 5, 8) *looks* good at first, since those three numbers can add and subtract to all numbers 0-9. BUT, if we imagine us trying to move 2 squares to the right, if we are in the 4th column, we'll see this is impossible: we'd have to move 8 right, 5 left, 1 left, in some order, which we don't have room for.

Now let's say we are starting on some square on the board, and suppose we start with 5 tiles. We know that after we've moved horizontally, we will need 3 tiles left over—one of the triplets just listed—in order to move anywhere we need to vertically. Therefore, it will have to be possible to move anywhere horizontally using any **2** of our 5 tiles.

In other words, we need 5 tiles such that

- any square in our row can be reached using at most **2** tiles; and
- the remaining tiles include one of the six triplets listed above.

There seems to be no elegant way of checking this *a priori*, so we will need to notice some patterns. Certainly we need two of our tiles to add up to 9, otherwise we could use up three right away. We also need a 1, since each of our triplets except (3, 4, 5) contains 1. [It can be shown quickly that there is no way to start with (3, 4, 5, __, __) and guarantee we are left with (3, 4, 5).] We actually need *two* 1's, because we could be forced to use up one of them right away. So we need (1, 1, *x*, *y*, *z*), containing at least one of our above triplets and such that $y + z = 9$. (We can't just have an 8, because if it *isn't* used right away, we'll be stuck with it and no triplet contains an 8.) There are several quintuples of this form, such as (1, 1, 7, 3, 6), (1, 1, *x*, 2, 7), (1, 1, 6, 2, 7), (1, 1, 2, 3, 6) … but for each of them, it is easy to come up with a horizontal number of squares such that afterwards the resulting tiles will *not* be one of our triplets. Therefore we need at least **6** tiles to be able to span the board from any starting square.

Appendix: Demonstration of the 3^n property where $n = 2$

We have two tiles – say, 4 and 5. Each tile has three uses: left, right, or none (L, R, 0). (Vertically would be up, down, or none). That means these are the only possible combinations (since order doesn't matter):

4	**5**	**Net result**
0	0	No move
0	L	5 left
0	R	5 right
L	0	4 left
L	L	9 left
L	R	1 right
R	0	4 right
R	L	1 left
R	R	9 right

These 9 moves do not even come close to covering all ten numbers between 0 and 9, let alone for both left and right. On the other hand, it can easily be shown that tiles 1, 3, and 5 *do* cover 0-9 in both directions (but the chart has 27 rows instead of just 9!).